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Avalanche distribution in the Feder and Feder model: Effects of quenched disorder

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The Feder and Feder earthquake model shows an integrated size distribution of events that decays as a power law when averaged over realizations with different initial conditions. The question remains as to what is the distribution for a single realization. Small amounts of quenched disorder can break the symmetries of the Feder and Feder model, introduce stochasticity in the dynamics, and allow for self-averaging. The introduction of weak frozen spatial disorder reveals a dynamical behavior very different from what is seen by ensemble averaging. The resulting integrated size distribution seems to be a function of the logarithm of the size, $P(n)\sim(\log_{10}n)^{-\nu}$. [S1063-651X(97)07103-1]

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We have seen in the last year or so a renewed interest in coupled integrate-and-fire (IAF) models of the type proposed by Feder and Feder (FF) [1] and Olami, Feder, and Christensen (OFC) [2]. This interest has sprung from a considerable effort to understand the origin of synchronization and to characterize the existence or not of self-organized criticality in dissipative models [3–8]. Although much progress has been made, many unresolved points remain regarding these two questions.

In this Brief Report, I study a modified version of the Feder and Feder model [1] introduced recently by Herz and Hopfield (HH) [6]. Although not a good representation for earthquakes [9–11], the FF coupled IAF oscillator has been considered as a possible candidate as a model of firing neurons where global synchronization and periodicity are thought to play an important role [6] and remains therefore a useful model to study. For periodic boundary conditions, HH showed that the model is periodic but with either purely local (involving a single site) or system-wide (all sites) events, depending on the parameters. In the more interesting case of open boundaries, however, the temporal behavior can be quite different. With this configuration, the FF model is known to rapidly converge to a cyclic behavior with periodically recurring avalanches happening all over the lattice (see Fig. 1). The specific avalanche size distribution depends only on the initial conditions. When an ensemble average is taken over different initial conditions, FF and HH found that the integrated size distribution for the avalanches follows a power law. The question as to what is the nature of this distribution for a single representation remains to be understood. One way to do this is to introduce very weak quenched spatial disorder and see whether this behavior is stable or not. Doing so, I have found that the integrated distribution for a single realization is not self-organized critical (SOC) but follows a law that decays much more slowly with increasing size of avalanches.

The FF rule was first proposed as a model describing the

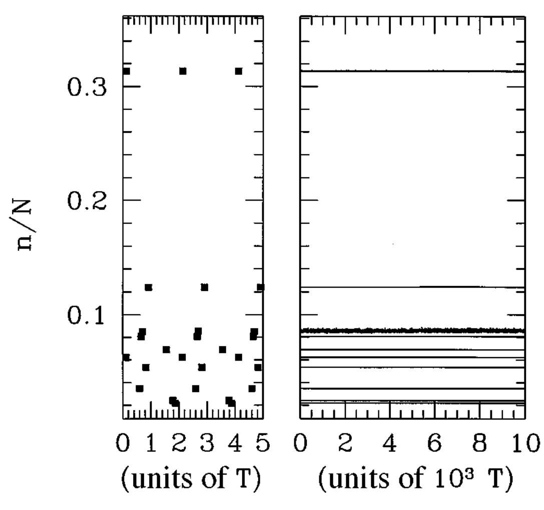

FIG. 1. Time evolution of the event size for the FF coupled-map model on a $L=225$ lattice with $\alpha=0.20$ and $\delta=0$. Only events of size larger than 1000 are shown. The time is scaled by the period T defined in the text and the amplitude by the total number of sites N. The left-hand part of the graph underlines the discrete nature of these avalanches, which happen one *after* the other; the right-hand part shows the long-time stability of this sequence of avalanches.

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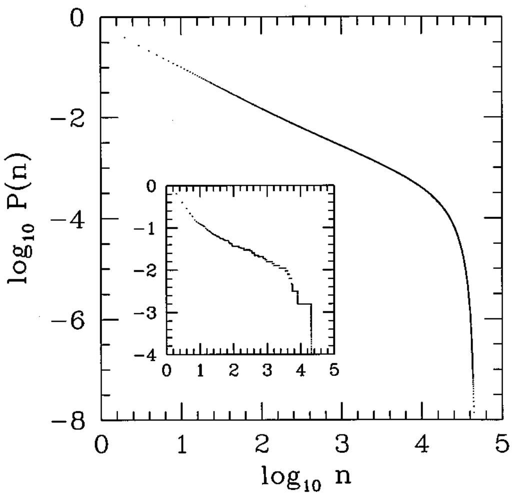

FIG. 2. Logarithm of the integrated probability of finding an avalanche of at least size n as a function of the logarithm n. Results obtained by an ensemble average of 100 different initial conditions for the same set of parameter as in Fig. 1. Inset: Same plot for the time evolution of Fig. 1 (single realization).

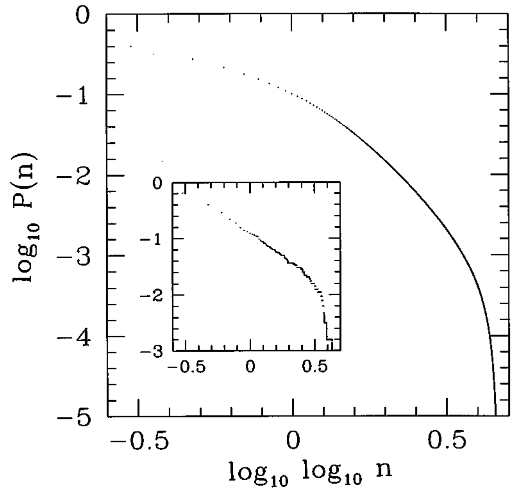

FIG. 3. Same as in the previous figure except that the scale is now log-double-log.

stick-slip motion of blocks covered with sandpaper and sliding on carpet [1], itself a simplistic representation of single-fault motion. At each site of a square lattice, a stress function is defined $F_i(t)$, which evolves linearly in time, under a sliding force,

$$\frac{dF_i(t)}{dt}=1. \tag{1}$$

When the stress at a site i reaches a threshold τ_c, taken to be 1 for simplicity, it is set to 0,

$$F_i(t)\rightarrow F_i'(t)=0, \tag{2}$$

and a fraction of the stress, $\Delta_i<1/4$, is redistributed to the nearest neighbor of i [6],

$$F_j(t)\rightarrow F_j'(t)=F_j(t)+\Delta_i. \tag{3}$$

The last two steps are repeated until the stress at every site is below the threshold τ_c. One can view this process as describing the blocks slipping and releasing their stress on their neighbors. Once all F_i's are within the allowed band of values, time is restarted. The size of an event, called avalanche by analogy with the sandpile models, is given by the number of sites triggered at a single time.

In previous works regarding this model, the redistribution factor Δ_i was taken to be site independent. Here, a small amount of quenched disorder is introduced as a zero-mean perturbation on a constant α:

$$\Delta_i=\alpha+\beta_i, \tag{4}$$

with β_i, a random variable taken in the interval $[-\delta,\delta]$. Small δ's ($\leqslant 0.020$) were generally used although larger values are also briefly discussed. A similar disorder was implemented on the OFC rule where it was found to affect significantly the dynamical behavior [8].

At $t=0$, the stress F_i is drawn from a linear distribution in the interval $[0,1)$ such that two sites never have the same stress simultaneously. This ensures that an event is always triggered from a single site. The mode of propagation of the avalanche from this initial site is not uniquely determined by the rule itself and must be chosen from physical (or numerical) considerations. The most commonly used algorithm is a radial move from the center. This approach was retained by Olami *et al.* [2], Grassberger [12], Herz and Hopfield [6], and others. The slowness of the algorithms used in the first studies of these models was such that only very small lattices with sides between $L=30$ and 50 could be simulated, making it difficult to identify size effects. With the introduction of better algorithms, however, Grassberger [12] could study square lattices with sides of length up to $L=1000$, and confirm and extend the results presented in [2]. This technique, which I have used here, improves the traditional approach for the problem by (1) keeping the stress F_i time independent, letting instead the threshold change with $d\tau(t)/dt=-1$, and (2) subdividing the lattice into many blocks, therefore increasing the speed of state reordering after an avalanche. The reader is referred to [12] for more details.

Figure 1 presents the temporal evolution of a typical realization for this model on a lattice of size 225^2 with $\alpha=0.20$, open boundaries, and no quenched disorder. Only events involving more than 1000 sites are plotted there. In the left panel, we plot a short time sequence of recurring avalanches. In the right panel, we show the long time behavior of this periodic sequence. Most of these events come back in exactly the same way time after time, producing *plateaus* in the time evolution. Some, however, fluctuate slightly but still remain very stable over long time sequences. The unit of time chosen is the natural return period for earthquake-type models with periodic boundary conditions, $T=1-4\alpha$, as was shown by Middleton and Tang [13] for the OFC coupled map and by HH [6] for a larger class of models, including FF. With open boundaries, the period of events settles at $P=1-3\alpha$; i.e., it is controlled by the boundary sites. As we shall see, this periodicity does not survive the introduction of quenched disorder. From Fig. 1, it

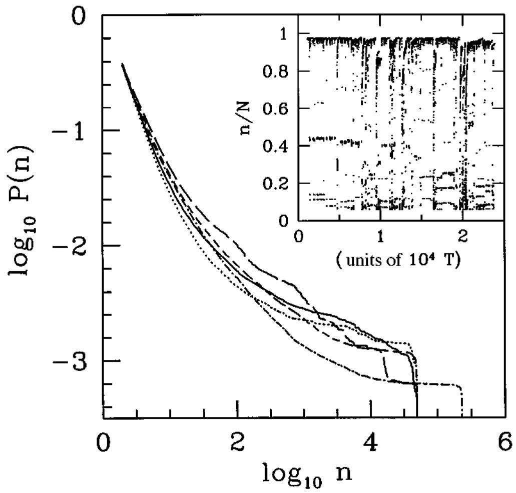

FIG. 4. Logarithm of the integrated size-probability distribution function as a function of the logarithm of the size for $L=225$ lattices with $\alpha=0.10$ and $\delta=0.001$ (solid line), $\delta=0.005$ (short dashes), $\delta=0.20$ (long dashes), and $\alpha=0.20$ with $\delta=0.002$ (dots) as well as for a $L=484$ lattice with $\alpha=0.20$ with $\delta=0.002$ (dot dashes). Inset: time evolution of the event for a $L=225$ lattice with $\Delta_i=0.20\pm0.0005$. N is the total number of sites on the lattice and T is the period defined in the text.

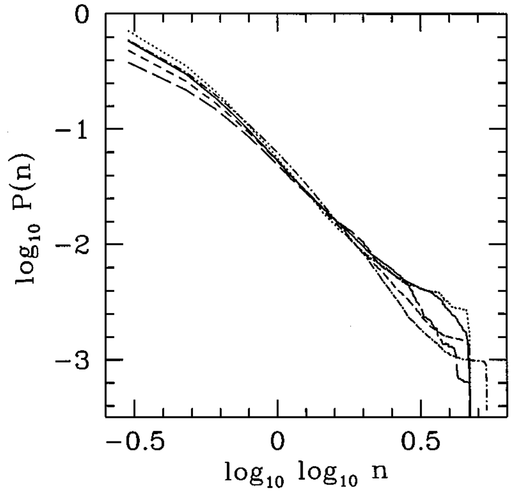

FIG. 5. Same as Fig. 4 except that the scale is now log-double–log.

can also be noted that in the absence of quenched disorder, only very small noise remains present in the time series. Although the system stays periodic for long stretches of time, some instabilities can build up and change the configuration of events [6] but their time scale is too long to be studied numerically.

Because of the quasifrozen temporal behavior of this nondisordered IAF model, self-averaging does not take place. In order to characterize the distribution, HH and FF took an ensemble average over the results of many realizations with initial conditions. Such an average produces an integrated size distribution of events that follows a power law (see Fig. 2). (This power-law behavior should not be seen as self-organized critical since it comes from the average of different systems, not self-averaging.) This average distribution, however, does not tell us much about the fundamental dynamics of a *single* realization.

If we look at such a quantity (inset of Fig. 2), it appears clear that this distribution does *not* follow a power law and that the weight is shifted towards larger avalanches. It turns out that this distribution is well approximated by a straight line in a log–double-log plot (Fig. 3). It is clear from this figure that the ensemble averaged distribution does not provide a proper understanding as to the real nature of the event distribution for a single realization.

To understand the statistics of event-size distribution for a given realization, it is useful to introduce some quenched disorder that should break the symmetries of the FF interactions, destabilize the plateaus, and produce a stochasticity similar to what is found in the OFC model [14]. The justification for the addition of disorder is twofold: First, by introducing disorder in the problem, it should no longer be necessary to perform an ensemble average, allowing us to study the statistics of a single fault at a time. Second, it is a test of the stability of the results obtained for the FF rule under small spatial perturbations; in order to be considered as a reasonable substitution for a macroscopic system like neurons or earthquakes, it is necessary that the model be stable under local variations.

As was shown in previous works [5,8,9,11,15], the behavior of these models can be strongly affected by even very small amounts of disorder. With the introduction of slight frozen disorder in the rule, all temporal behavior indeed becomes stochastic and self-averaging. The plateaus that are seen in the standard FF model become unstable and do not survive. However, and contrary to what could have been expected, with very small disorder the *integrated* probability distribution takes a well-defined shape that is in agreement with the one displayed by a single realization without any frozen disorder. It can be fitted by an unusual law favoring very large avalanches:

$$P(n) \sim \frac{1}{(\log_{10} n)^{\nu}}. \tag{5}$$

As for the OFC case [8], disorder seems to help larger events to take place. There is no full synchronization here, but the decay in the distribution is slower than a typical power law. Moreover, the nature of the distribution is not changed by the small disorder but rather made more prominent. As one increases the amplitude of the frozen disorder, Eq. (5) is no longer followed by the system.

The inset in Fig. 4 shows the time evolution for a $L=225$ lattice with $\alpha=0.100$ and $\delta=0.005$, i.e., with a very low degree of disorder. The generic time behavior is qualitatively different from what is seen in Fig. 1: (1) there are no stable plateaus anymore and the system appears stochastic. (2) Large avalanches, almost system wide in size, become now very common. This type of time evolution happens for all the length scales studied here, i.e., between $L=100$ and 484 and for a disorder amplitude smaller than about $\delta=0.01$. The smaller the δ, however, the longer it takes for the correlation between successive events to decay. It is therefore necessary to perform longer and longer runs in order to obtain meaningful statistics. This is in agreement with the fact that for zero disorder, the correlation time becomes extremely long, although it is not clear whether it diverges. Since there appears to be no transition between systems with small disorder and the perfect FF model, the insight gained with the formers should extend straightforwardly to the latter even though its symmetry makes this underlying collective nature more difficult to observe in the nondisordered limit.

As mentioned in the previous paragraph, there is no need for an ensemble average in the presence of quenched disorder. This disorder prevents the configuration from getting stuck in a single and periodic series of event, destroying the dependence on initial configuration that is seen in the FF model. The integrated probability distribution in Fig. 4 is shown for $L=225$ lattices, with different α's and δ's. We see smooth curves with no plateaus or jumps. The sharp turn at large n is a size effect and happens at about 95% of the total size of the system. By contrast, the largest events found in the $\delta=0$ systems remain usually small, often encompassing 30% or less of all sites, with the precise value depending on the particular initial configuration. It is clear, from the log-log representation, that the integrated probability distribution *cannot* be described as a simple power-law behavior, signature of SOC. The positive curvature of the distribution means that larger events are more favored over smaller ones than SOC would predict. As can be seen in Fig. 5, a log-double–log plot, the distribution is much better fitted by a function like Eq. (5), i.e., a power law of the logarithm of the event size $\log_{10}n$. In this representation, the distribution is a straight line over a full unit, and is not an artifact of the limited size of the system. Given the computational costs of simulation, extending significantly the regime would be prohibitive. Interestingly, the exponent ν appears to be relatively unaffected by the specific value of α and δ, as distributions obtained from various parameters all fall on top of each other. This effect could already be seen in the previous figure. The value of this exponent is $\nu\approx 2.6$. Its universality is not well understood at the moment especially given the fact that in a similar model [12], the SOC exponent depends sensitively on the exact value of α. Although I have shown results here for $L=225$ and 484, similar results are found for lattices of $L=100$. In this case, however, the system is not large enough to produce the system-wide avalanches seen in the inset of Fig. 4.

The power-law behavior in the logarithm of event size persists until some relatively large threshold in the amplitude of the quenched disorder. For a $L=484$ lattice with $\alpha=0.20$, for example, the threshold is at about $\delta=0.0025$. Beyond this value, the bias towards large events disappears and the distribution of avalanches is no longer well defined. There remains some tendency to organization, with some large avalanches but one does not find a simple law to describe this behavior. As a general rule, however, as the amplitude of the disorder increases, system-size avalanches become less frequent: the large degree of disorder prevents a system-wide synchronization by inhibiting an efficient propagation of the avalanche through the network.

In conclusion, I have shown that the introduction of frozen disorder in the modified Feder and Feder IAF model can lead to better understanding of what is really happening in the zero-disorder limit. In this limit, the lattice sticks into a quasistatic regime, which, with an ensemble average on different initial conditions, displays a power-law distribution of avalanches. Introducing a small amount of frozen site disorder, I show that this distribution is misleading and that the fundamental integrated distribution of events in the small disorder limit is more biased towards large avalanches and could be described by an unusual power law of the logarithm of the size. This result breaks down when disorder increases further.

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